Rotation of a Disk in Dilute Polymer Solutions

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Reduction in resistance to rotation of a disk in dilute, high-molecular-weight polymer solutions is calculated. The theoretical results, which are based on the assumption that the polymers increase the thickness of the viscous sublayer but leave the velocity defect law unaltered, are shown to be consistent with experimental data. The empirical coefficients that are related to the effect of the polymers on the thickness of the sublayer are found to be close to those measured in some pipe flows.

Introduction

TURBULENT shear flows of dilute high-molecular-weight polymers have been intensively investigated during the last decade because of the drag-reducing properties of such additives. Although the exact mechanism of drag reduction is not fully understood, it is apparent from velocity distribution measurements in pipe flows that the drag reduction is associated with an increased thickness of the "viscous sublayer" but that the structure of the flow outside the wall region is hardly affected by the polymers. It is also evident that drag reduction occurs only when the shear stress near the wall exceeds a critical value. Measurements in pipe flows have led Meyer[1] to suggest that the mean longitudinal velocity in dilute polymer solutions is described by the logarithmic equation

$$u/V^* = A \ln(zV^*/\nu) + B + \Delta u^+ \tag{1}$$

where

$$\Delta u^+ = 2AC \ln(V^*/V^*_{crit}) \tag{2}$$

z = the distance from the wall; C = a concentration dependent parameter (in Meyer's original paper the parameter $\alpha = 2AC \ln 10$ was used rather than C); V^* = the shear velocity, $V^* = (\tau w/\rho)^{1/2}$; V_{crit}^* = the shear velocity at the onset of drag reduction; and A,B = the constants used in the Newtonian case. Equation (1) is assumed to be valid for $V^* > V_{crit}^*$ whereas for $V^* < V_{crit}^*$, $\Delta u^+ \equiv 0$.

Integrating Eq. (1) over the pipe cross section, an approximate expression for the Darcy-Weisbach friction coefficient f is obtained;

$$(8/f)^{1/2} = A \ln Re(f)^{1/2} + \bar{B} + \Delta u^+ \tag{3}$$

where Re is the Reynolds number and $\bar{B} = B - 3A/2 - A \ln 4(2)^{1/2}$.

According to Eq. (2), the value of Δu^+ depends on two characteristic parameters of the polymer solution C and V_{crit}^*. Data collected by different investigators, with supposedly the same polymer solutions, do not always yield identical values of C or V_{crit}^*. One reason for the large scatter in the measured values of C and V_{crit}^* is undoubtedly the different characteristics of polymer blends sold under the same commercial name. Sometimes even molecular blends of a given polymer with identical average molecular weights have different properties, since the molecular weight distributions of the blends are not similar. In addition, the properties of a polymer solution might be changed by polymer degradation during the preparation of the solution or the experiment itself.

The success of models used to predict the values of C and V_{crit}^* theoretically have so far been limited. In 1966, Elata, Lehrer, and Kahanovitz[9] proposed that at the onset of drag reduction the characteristic time scale of the constant shear layer, ν/V_{crit}^{*2}, is of the same order of magnitude as that of the characteristic time of the polymer molecule t_1. The value of t_1 was assumed to be equal to the maximum relaxation time of the polymer molecule

$$t_1 = 6\mu_0[\eta]M/\pi^2RT \tag{4}$$

where μ_0 is the viscosity of the solvent, R the gas constant, T the absolute temperature, $[\eta]$ the intrinsic viscosity of the polymer solution, and M its molecular weight. Elata's model was used successfully to predict the onset of drag reduction in some experiments;[3] however, it was unsuccessful in many others.

In 1967 Virk[4] proposed characterizing the onset of drag reduction by the ratio of the radius of gyration of the polymer in solution, R_g, to the characteristic length ν/V_{crit}^*. The available data indicated that

$$V_{crit}^*R_g/\nu = 0.0075 \pm 0.00025 \tag{5}$$

Recent measurements by Patterson[5] showed, however, that the deviation of data from Virk's average value is even larger than originally observed. Moreover, Patterson's measurements with Polyox solutions showed that the value of $V_{crit}^*R_g/\nu$ depends on the concentration of the solution as well. Similar results were obtained by Whitsitt, Harrington, and Crawford[6] in solutions of Separan AP-30. Their measurements with Guar Gum, however, showed only a very mild dependence of V_{crit}^* on concentration.

No theoretical model is available to predict the value of C at different concentrations. The data support the intuitive argument that C should be proportional to the concentration for very dilute solutions.

The purpose of the present work is to examine the effect of polymer additives on the resistance to the rotation of a disk submerged in an unbounded polymer solution. An expression similar to Eq. (2) for the torque coefficient will be derived and compared with torque measurements by the authors and other investigators. The values of V_{crit}^* and C obtained from the analysis of the torque measurements will

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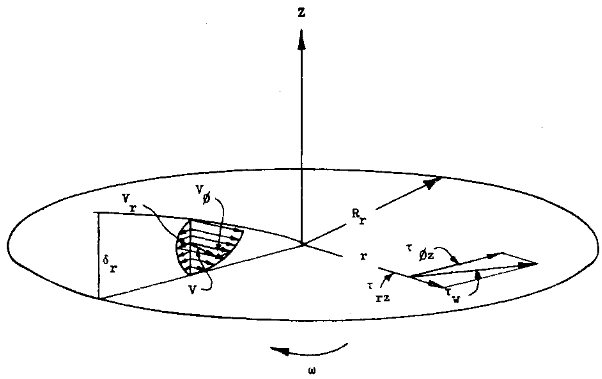

Fig. 1 Boundary layer on a rotating disk.

be compared with values obtained from measurements in pipe flow.

Analysis

The procedure of calculating the resistance to rotation of a disk in dilute polymer solutions is very similar to the one employed by Goldstein,[7] who analyzed the case of a disk rotating in a Newtonian fluid. In the following analysis, the logarithmic dimensionless radial and circumferential velocity profile used by Goldstein will be modified by adding the term Δu^+ proposed by Meyer. It should be noted that Δu^+ is not constant for a given flow since the shear stress varies along the radius of the disk. Near the center of the disk, where $V^* < V_{crit}^*$, there is no drag reduction and $\Delta u^+ \equiv 0$, whereas in the outer part of the disk, where $V^* > V_{crit}^*$, it is assumed that

$$\Delta u^+ = 2AC \ln(V^*/V_{crit}^*)$$

Drag reduction in case of a rotating disk will therefore occur when V^* at the edge of the disc exceeds the value of V_{crit}^*.

Figure 1 describes the cylindrical coordinate system adopted. The shear stress at the wall τ_w is divided into radial and circumferential components τ_{rz} and $\tau_{\phi z}$, which are related to the shear velocity by the equations

$$\tau_{rz} = a\rho V^{*2}/(1 + a^2)^{1/2} \tag{6}$$

$$\tau_{\phi z} = -\rho V^{*2}/(1 + a^2)^{1/2} \tag{7}$$

where

$$a = -\tau_{rz}/\tau_{\phi z} \tag{8}$$

The angle α between τ_w and the circumferential direction (see Fig. 1) is given by

$$\alpha = \tan^{-1} a \tag{9}$$

The radial and circumferential velocity components in a nonrotating coordinate system are denoted by V_r and V_ϕ. The circumferential velocity relative to the rotating disk $V_\phi{}^1$ is therefore

$$V_\phi{}^1 = V_\phi - r\omega$$

where ω is the angular velocity of the disk.

The absolute value of the velocity vector relative to the rotating disc will be denoted by V, so that

$$V^2 = V_r{}^2 + (V_\phi - r\omega)^2 \tag{10}$$

With the additional term Δu^+, Goldstein's assumptions give

$$V/V^* = A \ln zV^*/\nu + B + 2AC \ln(V^*/V_{crit}^*) \tag{11}$$

and

$$V_\phi{}^1 = V_\phi - r\omega = -V/(1 + a^2)^{1/2} \tag{12}$$

At the edge of the boundary layer $z = \delta$ and $V_\phi = 0$ so that

$$V(\delta) = r\omega(1 + a^2)^{1/2} \tag{13}$$

and thus,

$$V = r\omega(1 + a^2)^{1/2} + AV^* \ln(z/\delta) \tag{14}$$

From Eqs. (11) and (14) the value of δ is found to be

$$\delta = \nu e^Y (V_{crit}^*/V^*)^{2C}/B'V^* \tag{15}$$

where

$$Y = r\omega(1 + a^2)^{1/2}/AV^* \tag{16}$$

and

$$B' = e^{B/A} \tag{17}$$

Equation (15) may also be written as

$$\delta = Ke^Y(Y/r)^{1+2C} \tag{18}$$

where

$$K = \nu V_{crit}^{*2C}[A/\omega(1 + a^2)^{1/2}]^{1+2C}/B' \tag{19}$$

Equation (14) describes the absolute velocity defect law in both Newtonian fluids and dilute polymer solutions. Similarly, the equations for the radial and circumferential velocity components when expressed in terms of V^* and δ are identical to those used by Goldstein:

$$V_\phi = -AV^*(1 + a^2)^{-1/2} \ln(z/\delta) \tag{20}$$

and

$$V_r = ar\omega + aAV^*(1 + a^2)^{-1/2} \ln z/\delta \quad \text{for} \quad z < z_1 \tag{21}$$

$$V_r = -aAV^*(1 + a^2)^{-1/2} \ln z/\delta \quad \text{for} \quad z_1 \le z \le \delta \tag{22}$$

where z_1 is given by

$$\ln(z_1/\delta) = -r\omega(1 + a^2)^{1/2}/2AV^* \tag{23}$$

The radial and circumferential momentum integral equations for the boundary-layer flow near the disk are

$$-\frac{r}{\rho}\tau_{rz} = \frac{d}{dr}\left[r\int_0^\delta V_r{}^2 dz\right] - \int_0^\delta V_\phi{}^2 dz \tag{24}$$

and

$$-\frac{r}{\rho}\tau_{\phi z} = \frac{1}{r}\frac{d}{dr}\left[r^2\int_0^\delta V_\phi V_r dz\right] \tag{25}$$

Expressing τ_{rz} and $\tau_{\phi z}$ in terms of Y, using Eqs. (6, 7, and 16), and evaluating the previous integrals using Eqs. (20–23), the following equations are obtained:

$$-\frac{a(1 + a^2)^{1/2}}{A^2Y^2} r^3\omega^2 = \frac{d}{dr}\left[\frac{2a^2r^3\omega^2\delta}{Y^2}(1 - Ye^{-Y/2})\right] - \frac{2r^2\omega^2\delta}{Y^2} \tag{26}$$

and

$$\frac{(1 + a^2)^{1/2}}{A^2Y^2} r^3\omega^2 = \frac{1}{r}\frac{d}{dr}\left\{\frac{2ar^4\omega^2}{Y^2}\left[1 - \frac{1}{2}Ye^{-Y/2}(1 + 4/Y)\right]\right\} \tag{27}$$

Substituting in these equations the value of δ from Eq. (18) and multiplying by dr/dY, there results

$$\frac{d}{dY}\left[2a^2K\omega^2 r\left(\frac{r}{Y}\right)^{1-2C} e^Y(1 - Ye^{Y/2})\right] - 2K\omega^2\left(\frac{r}{Y}\right)^{1-2C}\frac{dr}{dY} = \frac{-a(1 + a^2)^{1/2}}{A^2Y^2} r^3\omega^2 \cdot \frac{dr}{dY} \tag{28}$$

and

$$\frac{d}{dY}\left\{2aK\omega^2r^2\left(\frac{r}{Y}\right)^{1-2C} e^Y\left[1-\frac{1}{2}Ye^{-Y/2}(1+4/Y)\right]\right\} = \frac{(1+a^2)^{1/2}}{A^2Y^2}\, r^4\omega^2\cdot\frac{dr}{dY} \quad (29)$$

Equations (24) and (25) are satisfied by the relations

$$r = \frac{K_0A^2}{a_0}(1+a_0^2)^{1/2}\left(\frac{Y}{r}\right)^{1+2C}\left[\left(1+\frac{A_1}{Y}+\frac{A_2}{Y^2}+\ldots\right)+ Ye^{-Y/2}\left(B_0+\frac{B_1}{Y}+\ldots\right)+\ldots\right] \quad (30)$$

and

$$a = a_0\left[\left(1+\frac{a_1}{Y}+\frac{a_2}{Y^2}+\ldots\right)+ Ye^{-Y/2}\left(b_0+\frac{b_1}{Y}+\ldots\right)+\ldots\right] \quad (31)$$

where

$$a_0 = \tfrac{1}{3} \quad (32)$$

and K_0 is the value of K [Eq. (19)] for $a = a_0$.

For large values of Y only the first terms of Eqs. (30) and (31) need be considered. Since according to Eq. (16) Y is inversely proportional to V^*, the asymptotic solution for $Y \to \infty$, which has been shown by Goldstein to be a good approximation in case of Newtonian fluids, is certainly justified in the case of polymer solutions which reduce the shear. In terms of the local Reynolds number

$$R_e = \omega r^2/\nu \quad (33)$$

one may write the asymptotic solution as

$$R_e = 3A^3[3AYV_{\text{crit}}^*/(10)^{1/2}\omega r]^{2C}Ye^Y/B' \quad (34)$$

For $C = 0$ the solution reduces to Goldstein's asymptotic solution:

$$R_e = 3A^3Ye^Y/B' \quad (35)$$

Equation (34) relates the local Reynolds number R_e and the parameter Y in the outer region of the disk $r > r_{\text{crit}}$, where $V^* > V_{\text{crit}}^*$. Equation (35) is valid for $r \le r_{\text{crit}}$. According to Eq. (16)

$$r_{\text{crit}} = 3(10)^{1/2}AYV_{\text{crit}}^*/10\omega \quad (36)$$

which indicates that Y, as described by Eqs. (34) and (35), is a continuous function of R_e and r.

The torque acting on the two sides of a rotating disc of radius R may be calculated from the momentum equation (25), which gives

$$M = -2\int_0^R 2\pi r^2\,\tau_{\phi z}\,dr = 4\pi\rho R^2\left[\int_0^\delta V_\phi V_r dz\right]_{r=R} \quad (37)$$

The second integral on the right hand side of Eq. (37) has already been evaluated and is given in the brackets of Eq. (27). Considering only the asymptotic value of the integral for large values of Y, one finds that

$$M = \rho\omega^2R^5[8\pi/3(10)^{1/2}](1/A^2Y^2) \quad (38)$$

and

$$K_m = \frac{M}{\frac{1}{2}\rho\omega^2R^5} = \frac{16\pi}{3(10)^{1/2}}\frac{1}{A^2Y^2} = \frac{(2.31)^2}{A^2Y^2} \quad (39)$$

From Eq. (34) it is found that

$$\frac{1}{(K_m)^{1/2}} = 2.31A\ \ln 10\left\{\log R_E\,(K_m)^{1/2} + \log\frac{2.31B'}{3A^3} + C\log\left[K_m\frac{5(10)^{1/2}}{24\pi}\frac{\omega^2R^2}{V_{\text{crit}}^{*2}}\right]\right\} \quad (40)$$

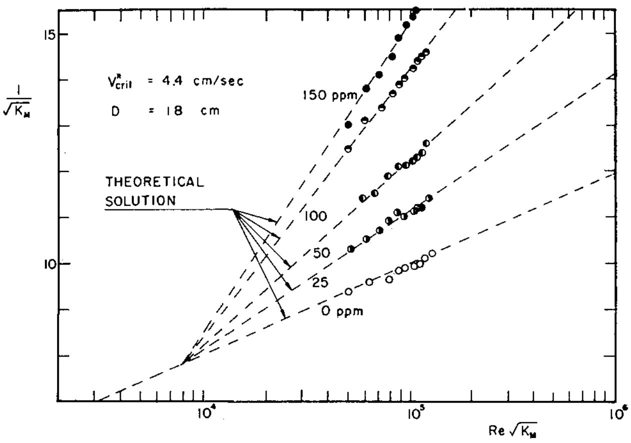

Fig. 2 Comparison of the theoretical solution and data of Miloh.[12]

where

$$R_E = \omega R^2/\nu \quad (41)$$

is the Reynolds number of the disk.

Analyzing all the available experimental data, Goldstein proposed $A = 1.97$ and $B = 6.53$ for disks rotating in Newtonian fluids. Using the same values, Eq. (40) becomes:

$$\frac{1}{(K_m)^{1/2}} = 1.97\log[R_E(K_m)^{1/2}] + 0.03 + 1.97C\log\left(0.21\,K_m\frac{\omega^2R^2}{V_{\text{crit}}^{*2}}\right) \quad (42)$$

The first terms of Eq. (42) give Goldstein's resistance formula for Newtonian fluids. Equation (42) may also be written in the form:

$$\frac{1}{(K_m)^{1/2}} = 1.97\log[R_E(K_m)^{1/2}]^{1+2C} + 0.03 + 1.97C\log\left(0.21\frac{\nu^2}{V_{\text{crit}}^{*2}R^2}\right) \quad (43)$$

The onset of drag reduction according to Eq. (42) is given by

$$0.21K_m\omega^2R^2/V_{\text{crit}}^{*2} = 1 \quad (44)$$

which corresponds, according to Eqs. (16) and (39) to the condition $V^*(R) = V_{\text{crit}}^*$.

Comparison with Experiments

It is convenient to compare the theoretical results with experimental data by plotting $1/(K_m)^{1/2}$ vs $\log R_E(K_m)^{1/2}$. Measurements by Miloh[12] of the torque acting on an 18-cm diam disk rotating in dilute solutions of Guar Gum J-2FP are shown in Fig. 2. Drag reduction appears to start at $R_E(K_m)^{1/2} = 8 \times 10^3$, which corresponds to a critical shear of $V_{\text{crit}}^* = 4.4$ cm/sec.

The same value of V_{crit}^* is obtained using Elata's onset criterion with $[\eta] = 12dl/gr$ (Ref. 13) and $M = 1.7 \times 10^6$ (Ref. 8), which give $t_1 = 5.7 \times 10^{-4}$ sec. It should be stressed, however, that the estimates of M and the measurements of $[\eta]$ given in the literature vary over a wide range.

Equation (42) is described in Fig. 2 by a set of straight lines converging at the critical values of $R_E(K_m)^{1/2}$. The data seems to follow closely the theoretical equation.

The measurements by Hoyt and Fabula,[9] who used a larger disk, $R = 22.85$ cm, and the same polymer, are shown in Fig. 3. The same value of V_{crit}^*, 4.4 cm/sec, was used to calculate the straight lines which describe the theoretical solution. Again, there is a reasonable agreement between the data and the theoretical solution.

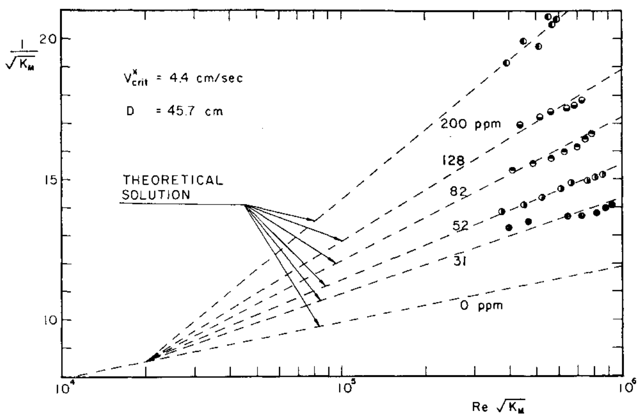

Fig. 3 Comparison of the theoretical solution with data of Hoyt and Fabula.[9]

It should be pointed out that the slopes of the straight lines in these figures, which are related to the value of C [see Eq. (43)], were arbitrarily adjusted to achieve a maximum correlation with the experimental data. The values of C determined in this manner are plotted vs concentration in Fig. 4. It is apparent from Fig. 4 that the two sets of data produce a continuous single-valued relationship between C and the concentration. It may be concluded, therefore, that the theoretical results are consistent with the available experimental data.

It is natural to inquire now whether the values of V_{crit}^* and C are the same as those found in pipe flows. Intuitively, one would expect them to be universal constants for any boundary-layer flow; on the other hand, they might be affected by the small differences in the detailed structure of the particular flow. In fact, it should be recalled that the coefficients A and B used by Goldstein to describe the resistance to the rotating of a disk in Newtonian fluids are not equal to those being used for pipes and flat plates. (The values of $A = 2.5$ and $B = 5.5$ are usually used in pipe flow and flat plates, rather than $A = 1.97$ and $B = 6.53$ as suggested by Goldstein for rotating disks.) This difference is usually attributed to the three-dimensional nature of the disk flow as well as to the vibrations of disks rotating at high angular velocities. The error introduced by Goldstein's asymptotic solution near the center of the disk is not considered to be significant since the moments are proportional to the fifth power of the radius. On the other hand, it should be pointed out that Goldstein's solution, and the present solution, are based on some intuitive assumptions about the velocity distributions.

As discussed in the introduction, the reported values of V_{crit}^* and C in the literature are not the same. Unfortunately, there are no experiments with pipes and rotating discs with exactly the same polymer solution. Whitsitt et al.,[6] have proposed a value of $V_{crit}^* = 3.5$ cm/sec for a 50 ppm Guar Gum J-2FP solution. Pressure measurements by Richter[10] and Hung[11] with the same polymer indicate a beginning of drag reduction around $V_{crit}^* = 4.4$ cm/sec. Hung has also measured the velocity profiles in a 1-in. pipe flow of dilute Guar Gum J-2FP solutions. His data, presented in Fig. 5, follow approximately the logarithmic profile, although they show, as expected, a deviation of the data from the log law near the center of the pipe and also very close to the wall. Using the value of $V_{crit}^* = 4.4$ cm/sec, the authors have calculated C from the measured displacement Δu^+ of the velocity profiles in Fig. 5. The calculated values are shown in Fig. 4 together with some values calculated using Eq. (3) from Richter's pressure drop measurements. All the points in this figure are thus calculated using the same critical shear velocity. The deviation of the pipe data from the rotating disk data is of the same order of magnitude as the deviation of the values of A and B from the corresponding values in pipe flow.

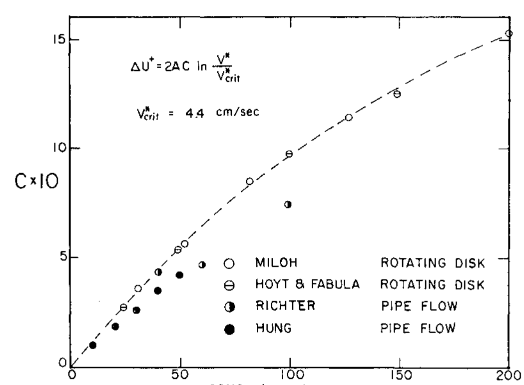

Fig. 4 Variations of the concentration dependent coefficient C in Guar Gum.

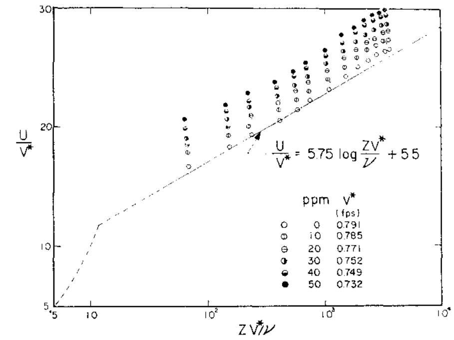

Fig. 5 Velocity measurements in pipe flow.[11]

Conclusions

A theoretical evaluation of the resistance to rotation of a disk in drag-reducing polymer solutions, based on Meyer's equation for pipe flow, is found to be consistent with the available experimental data. The value of the critical shear velocity that designates the onset of drag reduction is found to be close to those recorded for some pipe flows of the same polymer. The parameter C used in describing the effect of the polymer concentration on the magnitude of drag reduction in rotating disks is found to be only slightly larger.

These results support the view that the modified logarithmic velocity profile proposed for pipe flow is universally applicable in turbulent boundary-layer flows of dilute polymer solutions, as is the corresponding universal logarithmic velocity profile for Newtonian fluids.

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